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Absence of glassy behaviour in the deterministic spherical and *XY* models

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Abstract. We consider the infinite-range spin models with Hamiltonian $H = \sum_{i,j=1}^{N} J_{i,j}\sigma_i\sigma_j$, where J is the quantization of a map of the torus. Although deterministic, these models are known to exhibit glassy behaviour. We show, through explicit computation of the Gibbs free energy, that unlike the random case this behaviour disappears in the corresponding spherical and continuous XY models. The only minimum of the Gibbs free energy is indeed the trivial one, even though the ground state is highly degenerate.

1. Introduction

Various classes of infinite-range, deterministic Ising spin models which reproduce at least some of the 'glassy' properties of the random models have been introduced in the last few years [1–6]. It has been conjectured [4] that, unlike the random case, where it has been proved that the long-range spherical model admits a 'glassy' phase transition [7] (see also [8] for a review), the discrete nature of the spin variables is in this case a necessary condition to generate complex thermodynamic behaviour. The numerical analysis indeed shows [4] that this is the case for the fully frustrated Ising model on a hypercubic cell: this model is glassy and aging in the infinite-dimensional limit, but the numerical evidence also shows the disappearance of this behaviour in the corresponding compact and continuous XY case, i.e. when the spin variables are replaced by unimodular complex numbers. Moreover, similar results are conjectured [4] in the case of Heisenberg and spherical spins.

Here this conjecture is actually proved in the context of the deterministic models introduced in [2, 3, 6], which are characterized by two properties.

(i) The $N \times N$, infinite-range, non-translation-invariant coupling matrix $J^{(N)}$ defining the Hamiltonian

$$H^{(N)} = -\frac{1}{2}\sum_{i,j=1}^{N} J_{i,j}^{(N)}\sigma_i\sigma_j \tag{1.1}$$

coincides with (the real or imaginary part of) the unitary propagator quantizing the discrete dynamics generated by a symplectic matrix with integer coefficients

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad a, b, c, d \in \mathbb{Z}, ad - bc = 1$$

acting as a Hamiltonian map over the 2-torus $\mathbb{T}^2$. We recall that the operator quantizing a Hamiltonian map of the torus is a $N \times N$ unitary matrix [10, 11], N being the inverse of the

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Planck constant†: therefore, in this context the thermodynamic limit $N \to \infty$ is formally equivalent to the classical limit.

The case of [1–3], where glassy behaviour has been detected in numerical simulations, corresponds to the quantization of the unit symplectic matrix

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Here the coupling matrix J turns out to coincide with (twice the uppermost left block of) the discrete sine (cosine) Fourier transform

$$J_{i,j}^{(N)} = \frac{2}{\sqrt{2N+1}} \sin\left(\frac{2\pi i j}{2N+1}\right) \qquad i, j = 1, \dots, N. \tag{1.2}$$

The case of [6], where the existence of a 'glassy' critical point can be proved, corresponds instead to models whose coupling matrices are defined by the quantization of hyperbolic maps over $\mathbb{T}^2$ of the form

$$A = \begin{pmatrix} 2g & 1, \\ 4g^2 - 1, & 2g \end{pmatrix} \qquad g \in \mathbb{Z}$$

rather than elliptic ones such as S. The corresponding discrete dynamical systems yield indeed the best known examples of chaotic behaviour, while S generates in contrast a periodic discrete dynamical system of period four. The quantization of A is [10, 11, 13] the unitary $N \times N$ matrix

$$V(A)_{jk}^{(N)} = C_N \frac{1}{\sqrt{N}} \exp \frac{2\pi \mathrm{i}}{N}(gj^2 - jk + gk^2) \tag{1.3}$$

with $|C_N| = 1$ so that the models considered in [6] are defined by the Hamiltonians $H_A^{(N)}(\sigma) = \sum_{j,k} J(A)_{jk}^{(N)} \sigma_j \sigma_k$ with $J(A)^{(N)} = \mathrm{Re}\,[V(A)^{(N)}]$, i.e.

$$J(A)_{jk}^{(N)} = C_N \frac{1}{\sqrt{N}} \cos \frac{2\pi}{N}(gj^2 - jk + gk^2). \tag{1.4}$$

(ii) The ground state of both models is highly degenerate, depending on some arithmetical properties of the integer N. (In fact, for the first class of models the ground state can be explicitly computed [3] if $2N+1$ is prime with N odd, and its asympotic degeneracy along many other subsequences is proved in [9]. For the second class of models the proof is valid only in the present spherical case.) As has long been known [14], this suggests non-existence of the thermodynamic limit or, more precisely, the existence of different limits whether N goes to infinity along subsequences corresponding to degenerate ground states or not.

We thus consider here the Hamiltonians (1.2) and (1.4) (in fact, we replace (1.2) by any orthogonal $N \times N$ matrix) in the (continuous and compact, as in [4]) XY case as well as in the spherical one, namely

$$\sigma_i \in \mathbb{R} \qquad \sum_{i=1}^{N} \sigma_i^2 = N.$$

This system is known to have the same thermodynamic limit as the Heisenberg N-spin models [8]. As usual, these spherical models turn out to be exactly solvable, in the spherical and also in the (continuous and compact) XY case, in the sense that the Gibbs (i.e. magnetization-dependent) free energy can be computed in closed form as a function of the

† The physical intuition is that the phase space has volume 1, and can accommodate at most N quantum states of volume $\hbar$, so that $N\hbar = 1$.

limiting spectral measure. Its stationarity conditions yield the TAP (or mean-field) equations of the model [12], which are proved to admit only the trivial (i.e. zero magnetization) solution for any positive temperature, so that the system admits only the paramagnetic phase.

2. The models and the results

Before stating the results it is worth recalling some properties of the matrices defining our models.

(1) The choice of the form of A among the linear hyperbolic maps of the torus is motivated by the fact that if a, b, c, d are as above we clearly have† $\overline{V}(A^{(N)}) = \overline{V^{(N)}}^{\mathrm{T}}(A) = V^{(N)-1}(A)$, whence $\sigma(\mathrm{Re}\,[V(A)^{(N)}]) = \mathrm{Re}\,[\sigma(V(A)^{(N)})]$. Hence, denoting $\mathrm{e}^{\mathrm{i}\lambda_k^{(N)}}, 0 \leqslant \lambda_k^{(N)} < 2\pi, k = 1, \ldots, N$, the eigenvalues of $V(A)^{(N)}$, those of $J(A)^{(N)}$ are $p_k^{(N)} = \cos\lambda_k^{(N)}$.

(2) Let $E^{(N)}(x), \mathrm{d}\mu_N(x)$ be the spectral family and the spectral density of $V(A)^{(N)}$, respectively,

$$E^{(N)}(x) = \sum_{k:\lambda_k^N \leqslant x} \Pi_k^{(N)} \qquad \mathrm{d}\mu_N(x) = \frac{1}{N}\sum_{k=1}^{r(N)} M_k \delta(x - \lambda_k^{(N)}) \tag{2.1}$$

where the $r(N)$ distinct characteristic roots $\lambda_k^{(N)}$ have multiplicities $M_k^{(N)}$, the dimension of the corresponding orthogonal eigenprojections $\Pi_k^{(N)}$. $V(A)^{(N)}$ always admits the eigenvalue (1), and it has been proved in [18] that the weak* limit of the sequence $\mathrm{d}\mu_N(x)$, $N = 1, 2, \ldots,$ as $N \to \infty$ is the (normalized) Lebesgue measure $\mathrm{d}x$ on the circle S^1.

(3) By point (1) the eigenspaces of $J^{(N)}$ and $V^{(N)}$ coincide, so that by the spectral theorem

$$J(A)^{(N)} = \int_0^{2\pi} \cos x \,\mathrm{d}E^{(N)}(x). \tag{2.2}$$

(4) The orthogonal $N \times N$ matrices are clearly included in the above formalism: in this case one has $\lambda_k^{(N)} = 0$ or $\lambda_k^{(N)} = \pi$ so that $p_k^{(N)} = \pm 1$, $r(N) = 2$ and

$$E^{(N)}(x) = \begin{cases} 0 & 0 \leqslant x < \pi \\ \Pi_{\{-1\}}^{(N)} & \pi \leqslant x < 2\pi \\ \Pi_{\{1\}}^{(N)} + \Pi_{\{-1\}}^{(N)} & x = 2\pi \end{cases} \tag{2.3}$$

$$\mathrm{d}\mu_N(x) = \alpha(N)\delta(x) + (1 - \alpha(N))\delta(x - \pi) \tag{2.4}$$

where $\alpha(N) = M_1(N)/N$ is the relative proportion of the eigenvalue 1 or, equivalently, $N\alpha(N) = \dim \mathit{Ran}\, \Pi_{\{1\}}^{(N)}$.

(5) Let once more

$$E^{(N)}(x) = \sum_{k:p_k^N \leqslant x} \Pi_k^{(N)} \tag{2.5}$$

be the spectral family of the operator $J_{ij}^{(N)}$. We denote by $E(x)$ the limiting (orthogonal) projection operator valued measure on ℓ^2 of $E^{(N)}(x)$, supported on $[0, 2\pi[$, and J the

† $\overline{V}$ denotes the complex conjugate of V, and V^{T} its transpose.

corresponding set of bounded self-adjoint operators in ℓ^2:

$$J = \int_0^{2\pi} \cos x \, \mathrm{d}E(x). \tag{2.6}$$

(6) Any eigenvector corresponding to the eigenvalue 1 of $J^{(N)}$ clearly defines a ground state of $H^{(N)}$. The ground state is therefore degenerate of order $M_1^{(N)}$. We recall that the numerical evidence [16] indicates that for 'most' subsequences $\{N_K\}$, $M_1^{(N_k)}$ is bounded as $N_k \to \infty$. There are, however, subsequences for which $M_1^{(N_k)} \to \infty$ as $N_k \to \infty$.

Consider now the spherical model with Hamiltonian $H^{(N)}$, namely

$$H(\sigma)^{(N)} = -\sum_{i,j=1}^{N} J_{i,j}^{(N)} \sigma_i \sigma_j \qquad \sigma = (\sigma_1, \ldots, \sigma_N) \qquad \sum_{i=1}^{N} \sigma_I^2 = N.$$

Let $m_i = \langle \sigma_i \rangle$ be the magnetization at site i, and

$$q = \frac{1}{N} \sum_{i=1}^{N} m_i^2$$

the Edwards–Anderson order parameter. Rescale the magnetizations by $\sqrt{N}$, $m_i = \sqrt{N}\mu_i$, so that the Edwards–Anderson order parameter becomes $q = \sum_{i=1}^{\mathrm{N}} \mu_i^2$, $0 \leqslant q \leqslant 1$. Then the first result of this paper can be formulated as follows.

Proposition 2.1. Let μ belong to the open unit ball in ℓ^2, i.e. $q = \sum_{k=1}^{\infty} \mu_k^2 < 1$, and let $N \to \infty$. Let $\mathrm{d}\nu(x)$ denote $\mathrm{d}x/\pi$, where $\mathrm{d}x$ is the Lebesgue measure on $[0, 2\pi]$, or the pure-point measure $\alpha\delta(x) + (1-\alpha)\delta(x-\pi)$, $0 < \alpha < 1$. Then

(1) the condition

$$\int_0^{\pi} \frac{\mathrm{d}\nu(x)}{2\lambda - \beta \cos x} = 1 \tag{2.7}$$

implicitly defines a smooth function $\beta \mapsto \lambda(\beta) > \beta/2$ on $[0, +\infty[$;

(2) for $\beta \in [0, +\infty[$ the specific Gibbs free energy $\phi(\mu, \beta) = \Phi(\mu, \beta)/N$ has the following limit

$$-\beta\phi(\mu, \beta) = \frac{1}{2}(\ln[2\pi(1-q)] + 1) + \frac{\beta}{2}\langle J\mu, \mu\rangle + G(\beta(1-q)) \tag{2.8}$$

where

$$G(\beta) = \frac{1}{2}(2\lambda(\beta) - 1) - \frac{1}{2}\int_0^{\pi} \ln(2\lambda - \beta \cos x)\, \mathrm{d}\nu(x). \tag{2.9}$$

Remark. The function G is smooth for $q < 1$ because $\lambda > \beta/2$. Hence the free energy (2.8) is a family (indexed by β) of continuous functionals inside any closed ball or radius <1 in ℓ^2. The operator J is likewise continous in ℓ^2.

Given the expression (2.8) for the free energy the pure magnetization states are defined by its local minima; the stationarity conditions of ϕ

$$\frac{\mu_i}{1-q} + 2\beta G'(\beta(1-q))\mu_i - \beta(J\mu)_i = 0 \tag{2.10}$$

represent the TAP equations of the model. There exist phases other than the paramagnetic one if and only if the TAP equations (2.10) admit at least a solution other than the trivial solution $\mu = 0$. Hence the critical temperature for a phase transition, if any, is given by their linearization near $q = 0$:

$$\mu_i[1 + 2\beta G'(\beta) - \beta] = 0. \tag{2.11}$$

We can thus formulate the main result of this paper.

Proposition 2.2. Let $\beta > 0$. Then the TAP equations (2.10) admit only the trivial solution $\mu = 0$.

Remarks.

(1) At $T = 0$ the configuration of the system must a ground state, at which $\sigma_i = \langle\sigma_i\rangle = m_i$ so that $q = 1$. The ground states thus lie at the boundary $q = 1$ of the unit ball in ℓ^2. Hence, even though the system can lie in different magnetization states at $T = 0$, none of them generates long-range order, and the degeneracy does not affect the existence of the thermodynamic limit.

(2) The pure magnetization states at zero temperature are, however, 'glassy', in the sense that they are neither ferromagnetic nor antiferromagnetic: namely, according to the original definition of Edwards and Anderson, the average magnetization is zero but $q = 1$. It is indeed proved (see [9] for the orthogonal case, and [13] for the second one) that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mu_k^{(N)} = 0$$

if $\mu_k^{(N)}$ are the components of any normalized eigenvector of $J^{(N)}$ corresponding to the eigenvalue 1.

(3) The spherical model which we refer to is sometimes called the microcanonical model. The spherical constraint can be alternatively imposed on the mean, as done originally by Kosterlitz [7] (this represents the canonical or mean spherical model: for this issue see [8, section 3]). In the present non-translation invariant case the mean spherical model could *a priori* yield different thermodynamics in the presence of phase transitions; accordingly, we will check later that in the present situation the spherical and the mean spherical models yield the same free energies for all temperatures.

The result is even simpler for the (continuous and compact) XY model, defined once more by the Hamiltonian

$$H(\sigma)^{(N)} = -\sum_{i,j=1}^{N} J_{i,j}^{(N)}\sigma_i\sigma_j \qquad \sigma = (\sigma_1, \ldots, \sigma_N)$$

where now $\sigma_i = \mathrm{e}^{\mathrm{i}\theta_i}$, $0 \leqslant \theta_i < 2\pi$, $i = 1, \ldots, N$. The fact that the only stationary point of the Gibbs free energy is the trivial one takes the following form.

Proposition 2.3. Let h_k and m_k, $k = 1, \ldots, N$, be the magnetic field and the magnetization at site k, respectively, $h = (h_1, \ldots, h_N)$, $m = (m_1, \ldots, m_N)$. Let $Q^{(N)}$ be the orthogonal matrix diagonalizing $J^{(N)}$, and $G^{(N)}(m, \beta)$ be the Gibbs free energy. Then there exists a monotonic continuous function $x \mapsto f(x)$ on $]-\infty, \infty[$, vanishing linearly at $x = 0$, such that

$$\frac{\partial G^{(N)}}{\partial m_k} = 0, k = 1, \ldots, N \qquad \Longleftrightarrow \qquad (Q^{(N)}F)(m) = 0 \tag{2.12}$$

where $F = (f, \ldots, f)$.

3. Solution of the model and proof of the results

We first compute, by standard methods (see e.g. [8, section 3.4]) the Helmholtz free energy $F_N(\beta)$ for the (microcanonical) spherical model with the Hamiltonian $H^{(N)}$ in the (site-dependent) magnetic field $h = (h_1, \ldots, h_N)$. By definition the partition function $Z_N(\beta)$ at external, site-dependent magnetic field $h = (h_1, \ldots, h_N)$ is

$$Z_N(\beta, h) = \int_{\Omega_N} \exp[\beta\langle J^{(N)}\sigma, \sigma\rangle/2 + \beta\langle h, \sigma\rangle] = \prod_{i=1}^{N} \int_{-\infty}^{+\infty} \mathrm{d}\sigma_i \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}$$
$$\times \exp\left[\lambda\left(N - \sum_i \sigma_i^2\right) + \beta/2 \sum_{ij} J_{ij}^{(N)}\sigma_i\sigma_j + \beta \sum_i h_i\sigma_i\right].$$

Here Ω_N denotes the sphere $\sum_i \sigma_i^2 = N$, $c > 0$ and the equality follows (see e.g. [17]) by the well known δ function representation

$$\delta(x - a) = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^{\lambda(x-a)}\, \mathrm{d}\lambda.$$

Let $J_D^{(N)}$ be the diagonal form of $J^{(N)}$: $J_D^{(N)} = U^{-1}J^{(N)}U$, where $U = U^{(N)}$ is the $N \times N$ unitary matrix whose columns are eigenvectors of $J^{(N)}$. Denote once more p_k^N the eigenvalues of $J^{(N)}$. Denote furthermore

$$S = U\sigma \qquad \langle h, \sigma\rangle = \langle H, S\rangle \tag{3.1}$$

where

$$\sum_k H_k S_k = \sum_i h_i \sum_k U_{ik} S_k = \sum_{ki} U_{ik} h_i S_k \Longrightarrow H_k = \sum_i U_{ik} h_i \Longrightarrow H = U^* h = U^{-1} h. \tag{3.2}$$

One has, therefore, assuming $c > \beta p_k^N/2, k = 1, \ldots, N$, which implies uniform convergence of the integrals and hence their interchangeability

$$Z_N(\beta, h) = \prod_{k=1}^{N} \int_{-\infty}^{+\infty} \mathrm{d}S_k \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}} \exp\left[\lambda\left(N - \sum_k S_k^2\right) + \beta/2 \sum_k p_k^N S_k^2 + \beta \sum_k H_k S_k\right]$$
$$= \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\, \mathrm{e}^{N\lambda} \prod_{k=1}^{N} \int_{-\infty}^{+\infty} \mathrm{d}S_k \exp[-(\lambda - \beta p_k^N/2)S_k^2 + \beta H_k S_k].$$

Therefore, with $A(\lambda, \beta) = \sqrt{\lambda - \beta p_k^N/2}$

$$Z_N(\beta, h) = \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\, \mathrm{e}^{N\lambda} \prod_{k=1}^{N} \int_{-\infty}^{+\infty} \mathrm{d}S_k \exp\left[-\left(A(\lambda, \beta)S_k - \frac{\beta H_k}{2A(\lambda, \beta)}\right)^2\right.$$
$$\left. + \frac{\beta^2 H_k^2}{4A(\lambda, \beta)^2}\right] = \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\, \mathrm{e}^{N\lambda} \prod_{k=1}^{N} \left(\frac{\pi}{\lambda - \beta p_k/2}\right)^{1/2} \exp\left[\frac{\beta^2 H_k^2}{4(\lambda - \beta p_k/2)}\right]$$
$$= \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}} \exp\left[N G_N(\lambda, h) + \frac{N}{2}(\ln 2\pi + 1)\right]$$

where

$$N G_N(\lambda, \beta, h) + \frac{N}{2}(\ln 2\pi + 1) = N\lambda + \frac{N}{2}\ln\pi - \frac{1}{2}\sum_{k=1}^{N} \ln\left(\lambda - \frac{\beta}{2}p_k^N\right) + \frac{1}{2}\sum_{k=1}^{N} \frac{\beta^2 H_k^2}{2\lambda - \beta p_k^N}$$
$$= N\lambda + \frac{N}{2}\ln\pi + \frac{N}{2}\ln 2 - \frac{1}{2}\ln\det(2\lambda I - \beta J_D^{(N)})$$
$$+ \frac{\beta^2}{2}\langle(2\lambda I - \beta J_D^{(N)})^{-1} H, H\rangle.$$

Hence

$$G_N(\lambda,\beta,h)=\frac{1}{2}(2\lambda-1)-\frac{1}{2N}\ln\det(2\lambda I-\beta J^{(N)})+\frac{\beta^2}{2N}\langle(2\lambda I-\beta J^{(N)})^{-1}h,h\rangle. \tag{3.3}$$

We proceed now to evaluate the last integral through the saddle-point method for $N\to\infty$. First note that the limit of $G_N(\lambda,h)$ as $N\to\infty$ is

$$G(\lambda,\beta,h)=\frac{1}{2}(2\lambda-1)-\frac{1}{2}\int_0^\pi\ln(2\lambda-\beta\cos x)\,\mathrm{d}\nu(x)+\frac{\beta^2}{2}\int_0^\pi\frac{\mathrm{d}\nu_h(x)}{(2\lambda-\beta\cos x)} \tag{3.4}$$

where

$$\nu_h(x)=\lim_{N\to\infty}\frac{1}{N}\sum_{k:p_k\leqslant x}\langle h,\Pi_k h\rangle.$$

To apply the saddle-point method we need a preliminary result.

Lemma 3.1. There is $h_0>0$ such that the equation (2.7)

$$\mathcal{F}(\lambda,h,\beta)\equiv\int_0^\pi\frac{\mathrm{d}\nu(x)}{2\lambda-\beta\cos x}+\beta^2\int_0^\pi\frac{\mathrm{d}\nu_h(x)}{(2\lambda-\beta\cos x)^2}-1=0 \tag{3.5}$$

implicitly defines a smooth function $\lambda=f(\beta,h)$ on $[0,+\infty[\times[0,h_0]$ such that $\lambda(0,0)=\frac{1}{2}$.

Proof. Since we may assume that the measure $\mathrm{d}\nu(x)$ is normalized, the point $(\beta=0,h=0,\lambda=\frac{1}{2})$ fulfils the equation. Let now $h=0$ and consider separately the two cases $\mathrm{d}\nu=\mathrm{d}x/\pi$ and $\mathrm{d}\nu=\alpha\delta(x)+(1-\alpha)\delta(x-\pi)$. In the first case we have

$$\mathcal{F}(\lambda,0,\beta)=\frac{1}{\pi}\int_0^\pi\frac{\mathrm{d}x}{2\lambda-\beta\cos x}=\frac{1}{\sqrt{4\lambda^2-\beta^2}} \tag{3.6}$$

whence, solving for λ, $f(\beta,0)=\frac{1}{2}\sqrt{\beta^2+1}$ (the second root has to be discarded because smaller than $\beta/2$). Moreover,

$$\frac{\partial\mathcal{F}}{\partial\lambda}=-2\int_0^\pi\frac{\mathrm{d}\nu(x)}{(2\lambda-\beta\cos x)^2}-4\beta^2\int_0^\pi\frac{\mathrm{d}\nu_h(x)}{(2\lambda-\beta\cos x)^3}<0$$

for $|\beta|<2\lambda$ and h small and hence near $\lambda=f(\beta,0),h=0$. The result now follows by the implicit function theorem in this first case. In the second case we have instead

$$\mathcal{F}(\lambda,0,\beta)=\left[\frac{\alpha}{2\lambda-\beta}+\frac{1-\alpha}{2\lambda+\beta}\right] \tag{3.7}$$

whence

$$f(\beta,0)=\tfrac{1}{4}[1+\sqrt{4\beta^2+1-4\beta(1-\alpha)}]$$

and the proof is complete by the same argument verifying the conditions of the implicit function theorem. □

We can now evaluate the integral

$$I_N(\beta,h)=\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\,\mathrm{e}^{NG_N(\lambda,h)}$$

as $N\to\infty$ through the saddle-point method. As a function of $\lambda\in\mathbb{C}$, $G_N(\lambda,h)$ is clearly holomorphic for $\lambda\in\mathbb{C}\setminus]-\infty,\beta/2]$ and stationary for

$$\frac{1}{N}\sum_{k=1}^N\frac{1}{2\lambda-\beta p_k^N}+\beta^2\frac{1}{N}\sum_{k=1}^N\frac{\langle h,\Pi_k h\rangle}{(2\lambda-\beta p_k^N)^2}=1. \tag{3.8}$$

As is known, any real solution of this equation minimizes locally $\mathrm{Re}\,(G_N(\lambda))$; on the other hand, the limit of (3.8) is equation (3.5)

$$\int_0^\pi \frac{\mathrm{d}\nu(x)}{2\lambda - \beta\cos x} + \beta^2 \int_0^\pi \frac{\mathrm{d}\nu_h(x)}{(2\lambda - \beta\cos x)^2} = 1$$

whose solution has been discussed in lemma 3.1. Therefore, as $N \to \infty$

$$I_N(\beta, h) = \mathrm{e}^{NG(\beta,h)}(1 + \mathrm{O}(N^{-1}))$$

where $G(\beta, h) = \lim_{N\to\infty} G_N(f(\beta, h), \beta, h)$, namely, by (3.3) and (3.4)

$$G(\beta, h) = \frac{1}{2}(2f - 1) - \frac{1}{2}\int_0^\pi \ln(2f - \beta\cos x)\,\mathrm{d}\nu(x) + \frac{\beta^2}{2}\int_0^\pi \frac{\mathrm{d}\nu_h(x)}{(2f - \beta\cos x)} \tag{3.9}$$

where $f = f(\beta, h)$ is defined by lemma 3.1, formula (3.5). Hence, since

$$Z_N(\beta, h) = \exp[-\beta F_N(\beta, h)] = I_N(\beta, h)\exp\left[\frac{N}{2}(\ln 2\pi + 1)\right]$$

we obtain as $N \to \infty$ the following expression of the specific Helmholtz free energy $f_H(\beta, h) = \lim_{N\to\infty} F_N(\beta, h)/N$ at arbitrary magnetic field h:

$$-\beta f(\beta, h) = \tfrac{1}{2}(\ln 2\pi + 1) + G(\beta, h). \tag{3.10}$$

Proof of proposition 2.1. We proceed to compute the Gibbs free energy. To this end, we have to perform the Legendre transform of $-\beta f(\beta, h)$ with respect to h. Hence we first perform the Legendre transform of $-\beta F_N(\beta, h)$ and then take the $N \to \infty$ limit. The rescaling $m_i = \sqrt{N}\mu_i$ on the magnetizations generates the rescaling $h_i = \eta_i/\sqrt{N}$ on the dual variables, the site-dependent magnetic fields h_i. One has, therefore,

$$-\beta\Phi_N(\mu, \beta) = \max_{\eta\in\mathbb{R}^N}[\beta F_N(\beta, \eta) - \beta\langle\eta, \mu\rangle] = \beta F_N(\beta, \eta(\mu)) - \beta\langle\eta(\mu), \mu\rangle \tag{3.11}$$

where $\eta(\mu) : \mathbb{R}^N \to \mathbb{R}^N$ is obtained inverting $\mu_i = \partial F_N/\partial\eta_i, i = 1, \ldots, N$. The inversion is possible because the Hessian matrix of $-\beta F_N(\beta, \eta)$ with respect to h, given obviously by

$$\frac{\partial^2}{\partial\eta_i\partial\eta_j}\langle(2\lambda I - \beta J^{(N)})^{-1}\eta, \eta\rangle|_{\eta=0}$$

is positive definite since $\lambda > \beta/2$. Performing the Legendre transform we get

$$-\beta\Phi_N(\mu, \beta) = \frac{N}{2}[\ln 2\pi + 2\lambda] - \frac{1}{2}\ln\det(2\lambda I - \beta J^{(N)}) - \frac{1}{2}\sum_{i,j=1}^N \mu_i(2\lambda\delta_{ij} - \beta J_{ij}^{(N)})\mu_j \tag{3.12}$$

where now λ is determined by the stationarity condition

$$\frac{\partial\beta\Phi_N}{\partial\lambda} = 0. \tag{3.13}$$

Note that, equivalently,

$$-\beta\Phi_N(\mu, \beta) = \frac{N}{2}[\ln 2\pi + 2\lambda] - \frac{1}{2}\ln\det(2\lambda I - \beta J^{(N)}) - N\lambda q + \frac{\beta}{2}\sum_{i,j=1}^N J_{ij}^{(N)}\mu_i\mu_j \tag{3.14}$$

where once more

$$q = \sum_{k=1}^{N} \mu_k^2 = \frac{1}{N} \sum_{k=1}^{N} m_k^2$$

denotes the Edwards–Anderson order parameter. We reduce equation (3.13) to (3.8) and hence to (3.5) as $N \to \infty$ up to a rescaling of variables. To this end, first write (3.14) in the form

$$\beta \Phi_N = -\frac{N}{2}(\ln[2\pi(1-q)] + 1) - \frac{\beta}{2} \sum_{i,j=1}^{N} J_{ij}^{(N)} \mu_i \mu_j - N G_N^1(\lambda, \beta) \qquad (3.15)$$

where we have set

$$N G_N^1(\lambda, \beta) = \frac{N}{2}(2\lambda(1-q) - 1) - \frac{1}{2} \ln \det(2\lambda(1-q)I - \beta(1-q)J^{(N)}). \qquad (3.16)$$

Hence the stationarity condition (3.13) becomes $\mathrm{d}G_N^1(\lambda)/\mathrm{d}\lambda = 0$, that is, by (3.16)

$$\frac{1}{N} \sum_{k=1}^{N} \frac{1}{2\lambda(1-q) - \beta(1-q) p_k^N} = 1 \qquad (3.17)$$

which coincides with equation (3.8) at zero magnetic field up to the rescaling

$$\begin{cases} \lambda(1-q) \to \lambda \\ \beta(1-q) \to \beta. \end{cases} \qquad (3.18)$$

We can, therefore, take over to the present case the results of the above discussion: the (weak*) limit of (3.17) is equation (3.5) up to the rescaling (3.18). Therefore, as $N \to \infty$ the solution of (3.17) tends to $\lambda(1-q) = f(\beta(1-q))$, $f(\beta) \equiv f(\beta, 0)$. Now recall that, by the definitions (3.4), (3.9) and (3.16), $G_N^1(\lambda, \beta) = G_N(\lambda(1-q), \beta(1-q))$ where $G_N(\lambda, \beta) = G_N(\lambda, \beta, 0)$. Eliminating $\lambda(1-q)$ and setting $G(f(\beta), \beta) = G(\beta)$, $G(f(\beta(1-q)), \beta(1-q)) \equiv G(\beta(1-q))$ we finally obtain by (3.14) and (3.15) at the limit $N \to \infty$

$$-\beta\phi(\mu, \beta) = \frac{1}{2}(\ln[(2\pi(1-q)] + 1) + \frac{\beta}{2}\langle J\mu, \mu\rangle + G(\beta(1-q)).$$

This proves (2.8) and hence proposition 2.1. □

3.1. The mean spherical case

Let us now verify that the mean spherical model yields the same free energy. By definition (see e.g. [8, section 2]), the partition function is now

$$Z_N(\beta, h) = \int_{\mathbb{R}^N} \exp\left[\beta\langle J^{(N)}\sigma, \sigma\rangle/2 + \beta\langle h, \sigma\rangle - \frac{\beta z}{2} \sum_{i=1}^{N} (\sigma_i^2 - 1)\right] \prod_{i=1}^{N} \mathrm{d}\sigma_i$$

where the Lagrange multiplier z is this time to be determined by imposing the spherical constraint only in the mean, namely by the condition

$$\left\langle \sum_{i=1}^{N} \sigma_i^2 \right\rangle = N \qquad (3.19)$$

where $\langle\cdot\rangle$ denotes the (normalized) average over the above distribution. By diagonalization of $J^{(N)}$ and straightforward computation of the Gaussian integral we get, in the above notation,

$$
\begin{aligned}
Z_N(\beta, h, z) &= \mathrm{e}^{N\beta z/2}\left(\frac{2\pi}{\beta}\right)^{N/2}[\det(zI - J_D^{(N)})]^{-1/2}\exp\left[\frac{\beta}{2}\langle (zI - J_D^{(N)})^{-1}H, H\rangle\right] \\
&= \exp\left[NG_N^{(2)}(z, \beta, h) + \frac{N}{2}(\ln 2\pi + 1)\right]
\end{aligned}
$$

where

$$
G_N^{(2)}(z, \beta, h) = \frac{1}{2}(\beta z - 1) - \frac{1}{2}\ln\beta - \frac{1}{2N}\ln\det(zI - \beta J^{(N)}) + \frac{\beta}{2N}\langle (zI - \beta J^{(N)})^{-1}h, h\rangle.
$$

The limit of $G_N^{(2)}(z, \beta, h)$ as $N \to \infty$ is

$$
G^{(2)}(z, \beta h) = \frac{1}{2}(\beta z - 1) - \frac{1}{2}\ln\beta - \frac{1}{2}\int_0^\pi \ln(z - \cos x)\,\mathrm{d}\nu(x) + \frac{\beta}{2}\int_0^\pi \frac{\mathrm{d}\nu_h(x)}{z - \cos x}.
$$

The Lagrange multiplier z is determined by the condition

$$
\begin{aligned}
&Z_N^{-1}\int_{\mathbb{R}^N}\sum_{i=1}^N \sigma_i^2 \exp\left[\beta\langle J^{(N)}\sigma, \sigma\rangle/2 + \beta\langle h, \sigma\rangle - \frac{\beta z}{2}\sum_{i=1}^N(\sigma_i^2 - 1)\right]\prod_{i=1}^N \mathrm{d}\sigma_i - N = 0 \\
&\qquad \Longleftrightarrow \qquad \partial_z\left[-\frac{2}{\beta}\ln(Z_N)\right] = 0 \qquad \Longleftrightarrow \qquad \partial_z[G_N^{(2)}(z, \beta, h)] = 0.
\end{aligned}
$$

As $N \to \infty$ we get

$$
\frac{1}{2}\int_0^\pi \frac{\mathrm{d}\nu(x)}{z - \cos x} + \frac{\beta}{2}\int_0^\pi \frac{\mathrm{d}\nu_h(x)}{(z - \cos x)^2} - \frac{\beta}{2} = 0. \tag{3.20}
$$

Set $z = 2\lambda/\beta$. Then clearly z is a solution of (3.20) if and only if λ solves (3.5). To sum up, we obtain

$$
\begin{aligned}
G^{(2)}(z, \beta, h) \equiv G^{(2)}\left(\frac{2\lambda}{\beta}, \beta, h\right) &= \frac{1}{2}(2\lambda - 1) - \frac{1}{2}\ln\beta - \frac{1}{2} \\
&\quad \times \int_0^\pi \left[\ln\left(\frac{1}{\beta}\right) + \ln(2\lambda - \beta\cos x)\right]\mathrm{d}\nu(x) + \frac{\beta}{2}\int_0^\pi \frac{\beta\,\mathrm{d}\nu_h(x)}{2\lambda - \beta\cos x} \\
&= G(\lambda, \beta, h)
\end{aligned}
$$

and the subsequent discussion can be taken over to the present case without change.

Proof of proposition 2.2. By (2.8) the stationarity conditions of the specific Gibbs free energy with respect to the magnetizations are the TAP equations (2.10), rewritten here for convenience of exposition

$$
\frac{\mu_i}{1 - q} + 2\beta G'(\beta(1 - q))\mu_i - \beta(J\mu)_i = 0. \tag{3.21}
$$

The standard procedure (see e.g. [15]) to determine the highest critical temperature, if any, for the transition from the paramagnetic ($\mu_i = 0\ \forall i$) phase is first to linearize these equations near $q = 0$, i.e. to neglect all terms of order μ_i^l, $l \geqslant 2$. In this approximation we get the linearized TAP equations

$$
\mu_i(1 + 2\beta G'(\beta)) - \beta(J\mu)_i = 0. \tag{3.22}
$$

Furthermore, these equations are considered at size N, namely

$$
\mu_i(1 + 2\beta G'(\beta)) - \beta(J^{(N)}\mu)_i = 0 \qquad \mu = (\mu_1, \ldots, \mu_n) \tag{3.23}
$$

where, however, $G(\cdot)$ is computed at the thermodynamic limit $N \to \infty$. If μ^* is any eigenvector of $J^{(N)}$ corresponding to the eigenvalue 1 the equations (3.22) become

$$1 + 2\beta G'(\beta) - \beta = 0. \tag{3.24}$$

Let us first prove that in both cases this equation has no solutions. If $\mathrm{d}\nu = \mathrm{d}x/\pi$, by (3.9) (with $h = 0$) we get (recall that $\lambda = f(\beta)$)

$$G(\lambda, \beta) = \lambda - \frac{1}{2} - \frac{1}{2}\ln\left[2\lambda + \frac{\sqrt{4\lambda^2 - \beta^2}}{2}\right]. \tag{3.25}$$

Recalling that $\lambda = f(\beta) = \sqrt{1+\beta^2}/2$, (3.24) becomes

$$(1-\beta)(\sqrt{1+\beta^2} + 1) + \beta^2 = 0 \tag{3.26}$$

which clearly has no solutions because the left-hand side is positive on $[0, +\infty[$. In the case of $\mathrm{d}\nu = \alpha[\delta(x) + (1-\alpha)\delta(x-\pi)]$ we have

$$G(\lambda, \beta) = \lambda - \frac{1}{2} - \frac{\alpha}{2}\ln(2\lambda - \beta) - \frac{1-\alpha}{2}\ln(2\lambda + \beta). \tag{3.27}$$

Here the implicit equation (3.5) of lemma 3.1 clearly yields the function

$$f(\beta, \alpha, 0) = \tfrac{1}{4}[1 + \sqrt{4\beta^2 + 1 - 4\beta(1-2\alpha)}]$$

and (3.24) now becomes, after a straightforward computation,

$$1 + \beta\left[\frac{\alpha}{2f - \beta} - \frac{1-\alpha}{2f + \beta} - 1\right] = 0. \tag{3.28}$$

This equation has no solutions on $[0, +\infty[$ because the left-hand side is easily verified to be monotonically increasing in β for any $0 < \alpha \leqslant 1$ and is 1 for $\beta = 0$.

Let us now reduce the nonlinear equations (3.21) to the linear case just discussed. Setting $\xi = \beta(1-q)$ they become

$$\mu_i + 2\xi G'(\xi)\mu_i - \xi(J^{(N)}\mu)_i = 0 \tag{3.29}$$

or

$$[(1 + 2\xi G'(\xi))I - \xi J^{(N)}]\mu = 0$$

where I is the identity $N \times N$ matrix. This equation has a solution if and only if ξ is such that $1+2\xi G'(\xi)$ belongs to the spectrum of $\xi J^{(N)}$. This occurs only if $1+2\xi G'(\xi) - \lambda\xi = 0$ for some $-1 \leqslant \lambda \leqslant 1$, which is impossible as we have checked before. This completes the proof of proposition 2.2. $\square$

Let us now consider the following.

Proof of proposition 2.3. By definition the partition function is given by the following integral

$$Z_N(\beta, h) = \int_{\mathbb{R}^{2N}} \prod_{i=1}^{N} \exp\left[\frac{\beta}{2}\langle J^{(N)}\sigma, \sigma\rangle + \beta\langle h, \sigma\rangle\right]\delta(|\sigma_i| - 1)\,\mathrm{d}\sigma_i\,\mathrm{d}\overline{\sigma}_i.$$

As above, the change of variables (3.1) and (3.2) yields

$$Z_N(\beta, H) = \prod_{i=1}^{N}\int_{\mathbb{R}^2} \exp\left[\frac{\beta}{2}p_i|S_i|^2 + \beta H_i S_i\right]\delta(|S_i| - 1)\,\mathrm{d}S_i\,\mathrm{d}\overline{S}_i$$

whence, integrating in polar coordinates,

$$Z_N(\beta, H) = \exp \frac{\beta}{2} \sum_{k=1}^{N} p_k \prod_{k=1}^{N} I_0(\beta H_k)$$

where $I_0(x) = J_0(\mathrm{i}x)$ is the well known Bessel function of order zero. Consider the specific Helmholtz free energy $\beta f_N(\beta, H)$ as a function of the rotated field H,

$$\beta f_N(\beta, H) = \frac{1}{N} \ln Z_N(\beta, H).$$

One clearly has

$$\beta f_N(\beta, H) = \frac{\beta}{2N} \sum_{k=1}^{N} p_k^{(N)} + \frac{1}{N} \sum_{k=1}^{N} \ln I_0(\beta H_k).$$

Now set

$$M_l(H_l) = \frac{\partial \beta f_N(\beta, H)}{\partial H_l} \qquad l = 1, \ldots, N$$

(rotated magnetizations). Differentiating, we get

$$M_l(H_l) = \frac{\beta}{N} \frac{I_0'(\beta H_l)}{I_0(\beta H_l)} \qquad l = 1, \ldots, N$$

whence $M_l(H_l) \to 0$ as $H_l \to 0$ for any β. This means that the system is paramagnetic for any temperature. Since, moreover, the right-hand side is monotonic with respect to $H_l \in [0, +\infty[$, this last relation can be globally inverted to yield a smooth function $H_l = H_l(M_l), l = 1, \ldots, N$, vanishing only as $M_l \to 0$.

By definition the Legendre transform of $\beta f_N(\beta, H)$ is

$$\beta \phi_N(\beta, M) = \sum_{k=1}^{N} M_k H_k(M_k) - \frac{1}{N} \sum_{k=1}^{N} \ln I_0(\beta H_k(M_k))$$

and it is immediately verified that $\partial \beta \phi_N(\beta, M)/\partial M_l = H_l(M_l)$. The stationarity conditions for the Gibbs free energy, i.e. the TAP equations of the models, are therefore the conditions $H_l(M_l) = 0, l = 1, \ldots, N$, which admit only the trivial solution $M_l = 0$, $l = 1, \ldots, N$. Rotating back to the variables m and h we conclude the proof of the proposition. □

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